4 [9].—RICHARD P. BRENT, *Tables Concerning Irregularities in the Distribution of Primes and Twin Primes,* Computer Centre, Australian National University, Canberra, 1974, 11 computer sheets deposited in the UMT file.

These are the tables referred to repeatedly in Brent's paper [1]. The numbers $\pi(n)$, $\pi_2(n)$ and $B^*(n)$ and

$$r_i(n),\ s_i(n),\ R_i(n, n'),\ \rho_i(n, n')$$

for $i = 1, 2, 3$ are defined in [1]. They are listed in Table 1 for 533 values of n:

$$10^4\ (10^4)\ 10^6\ (10^5)\ 10^7\ (10^6)\ 10^8\ (10^7)\ 10^9\ (10^8)\ 10^{10}\ (10^9)\ 83 \cdot 10^9.$$

Table 2 (1 page long) lists n, $\pi_2(n)$, $B(n)$, and $B^*(n)$ with some auxiliary functions for

$$10^5\ (10^5)\ 10^6\ (10^6)\ 10^7\ (10^7)\ 10^8\ (10^8)\ 10^9\ (10^9)\ 10^{10}\ (10^{10})\ 8 \cdot 10^{10}:$$

The author indicates that he has much more detailed tables and is continuing to 10^{11}.

Section 3 of [1] ends with the same conclusion given earlier in our [2]: that the unpredictable fluctuations of $\pi_2(n)$ around the Hardy-Littlewood approximation makes it difficult to compute Brun's constant accurately. But his Fig. 3 allows for a posteriori judgment; although we do not know where $s_3(n)$ is going, we know where it's been! We see that Fröberg's low value at $\log_{10} n = 6.02$, our high value at $\log_{10} n = 7.51$ and Bohman's low value at $\log_{10} n = 9.30$ all correlate (inversely) with the peaks and valleys of Fig. 3. In fact, Fig. 3 between $\log_{10} n = 6.63$ and 7.19 gives a crude, distorted, upside-down version of our Fig. 1 [2] and $\log_{10} n$ between 7.19 and 7.51 continues with our Fig. 2. Thus, for Brun's constant, it does appear that $n = 8 \cdot 10^{10}$ is a good time to quit since $s_3(n)$ is then very small.

Concerning the negative peaks in Brent's Fig. 1 at $\log_{10} n = 8.04$ and 8.25, it would be nice to know when they are exceeded. As Brent is aware, if a likely n were known that is not too large, one could restart his tables of $r_i(n)$ and $s_i(n)$ for $i = 1, 2$ by computing a fiducial mark $\pi(n)$ by Lehmer's method.

D. S.

1. RICHARD P. BRENT, "Irregularities in the distribution of primes and twin primes," *Math. Comp.*, v. 29, 1975, pp. 43–56 (this issue).

2. DANIEL SHANKS & JOHN W. WRENCH, JR., "Brun's constant," *Math. Comp.*, v. 28, 1974, pp. 293–299; "Corrigendum", *ibid*, p. 1183.

5 [9].—CARL-ERIK FRÖBERG, *Kummer's Förmodan,* Lund University, 1973, 133 pages of computer output deposited in the UMT file.

The Kummer Sum

$$(1)\qquad S_p = \sum_{n=0}^{p-1} \exp(2\pi i n^3/p) = 1 + 2 \sum_{n=1}^{(p-1)/2} \cos(2\pi n^3/p)$$

for a prime $p \equiv 1 \pmod 3$ equals one of the three real roots of

$$x^3 = 3px + pA \tag{2}$$

where $4p = A^2 + 27B^2, A \equiv 1 \pmod 3$. On the basis of only the 45 primes $p < 500$, Kummer conjectured that S_p occurs as the minimum, median, or maximum root of (2) in the proportions: 1, 2, 3. Subsequent work of von Neumann [1] and Emma Lehmer [2] suggested that as $p \longrightarrow \infty$ there may be equidistribution instead, and Vinogradov once thought [3] that he had proven this.

Fröberg [4] computed S_p for the 8988 $p < 2 \cdot 10^5$ and found 2370, 2990, and 3628 solutions, respectively, with the maximal roots now down to 40.4%, the minimal roots up to 26.4% and the median roots remaining very close to 33%. There is deposited here a listing of these 8988 primes: p, A, B, S_p (to 6D), and an asterisk in the appropriate column labelled MIN, MED, MAX. S_p has rounding errors (example below) but this accuracy is not needed here since it suffices to know where S_p lies in the three intervals: $I_1 < -\sqrt{p} < I_2 < +\sqrt{p} < I_3$. Note also that it is unnecessary to compute A and B separately, since $A = (S_p^3 - 3pS_p)/p$.

After extrapolating the three empirical percentage functions $\%(P)$, for $p < P$, according to the proposed formulas

$$\%(P) = a + b \exp(-cP), \tag{3}$$

Fröberg conjectures that the asymptotic proportions are 4, 5, 6—that is, that the limiting percentages are $26\frac{2}{3}$, $33\frac{1}{3}$, and 40%, respectively. This reviewer is skeptical for two reasons: (A) No rationale, even heuristic, is given to support (3) and the exponential there tends to leave the purported asymptotic values a near his final empirical values at $P = 2 \cdot 10^5$. Whereas, any logarithmic function in place of (3) would make equidistribution more plausible. (B) If 4, 5, 6 are the true asymptotic proportions, it should be possible to find some reasonably simple heuristic argument that suggests these proportions. I know of none.

There are 51 cases here with $A > 0, B = 1$. Here the two smaller roots are nearly equal, being approximately $-\sqrt{p} \mp 1\frac{1}{2}$, while the largest root is nearly $+2\sqrt{p}$. If there is a difference in the ultimate proportion of MIN and MAX one might expect to see it here since the dissymmetry is maximized. One does not; there are 16, 18, and 17 cases, respectively. In the 53 cases with $A < 0, B = 1$, there is the opposite dissymmetry with the two larger roots close together near $\sqrt{p} \mp 1\frac{1}{2}$. One now finds 16, 19, 18 cases. (For more on the cyclic cubic fields with $B = 1$, see [5].) In the 74 cases here with $A = +1, -2, +4$ or -5, where the median root is $\approx -A/3$ while the extreme roots are $\approx \mp \sqrt{3p}$, one has the greatest symmetry. Here one finds 24, 21, and 29 cases. These are all small numbers but they seem to suggest equidistribution; certainly nothing here suggests that the MAX are 50% more numerous than the MIN. But if there is equidistribution, why are the MAX more common when p is small? A good, quantitative explanation is wanted.

$|S_p|$ is bounded below by 1/3. The smallest S_p here is one of the aforementioned $A = 1$, namely, $p = 170647, A = 1, B = 159, S_p = -0.3333334056$. (The table lists $S_p = -0.335414$ for this p, showing that four decimals are corrupted in adding up the 85 thousand cosines.) The existence of such small S_p illustrates the marked distinction between these cubic sums and the quadratic Gauss Sums with n^2 instead of n^3 in (1). Then, $|S_p| = \sqrt{p}$, as is well known. For other recent work, see Cassels [6] and the references cited there.

D. S.

1. J. v. NEUMANN & H. H. GOLDSTINE, "A numerical study of a conjecture of Kummer," *MTAC*, v. 7, 1953, pp. 133–134.

2. EMMA LEHMER, "On the location of Gauss sums," *MTAC*, v. 10, 1956, pp. 194–202.

3. A. I. VINOGRADOV, "On the cubic Gaussian sum," *Izv. Akad. Nauk SSSR Ser. Mat.*, v. 31, 1967, pp. 123–148. (Russian)

4. C.-E. FRÖBERG, "New results on the Kummer conjecture," *BIT*, v. 14, 1974, pp. 117–119.

5. DANIEL SHANKS, "The simplest cubic fields," *Math. Comp.*, v. 28, 1974, pp. 1137–1152.

6. J. W. S. CASSELS, "On Kummer sums," *Proc. London Math. Soc.*, v. 21, 1970, pp. 19–27.

6 [9].—MARIE NICOLE GRAS, "Méthodes et algorithmes pour le calcul numérique du nombre de classes et des unités des extensions cubiques cycliques de Q," Institut de mathématiques pures, Grenoble, 1972–1973. Tables 1–4.

For any product $m = p_1 \cdot p_2 \cdot \cdots \cdot p_n$ of distinct primes $p \equiv 1 \pmod 3$ there are 2^{n-1} distinct cyclic cubic fields of discriminant m^2 and for $m = 9 \cdot p_1 \cdot \cdots \cdot p_n$ there are 2^n such fields. Altogether there are 630 fields with $m < 4000$. Table 1 lists each such m with (A) its prime decomposition; (B) its appropriate representation $4m = a^2 + 27b^2$; (C) its class number h; and, in most cases, (D) $\mathrm{tr}(\epsilon)$ and $\mathrm{tr}(\epsilon^{-1})$. These latter integers give the equation

$$x^3 = \mathrm{tr}(\epsilon)x^2 - \mathrm{tr}(\epsilon^{-1})x + 1$$

satisfied by the fundamental units and having a discriminant m^2k^2 for some index $k \geqslant 1$. When $\mathrm{tr}(\epsilon)$ and $\mathrm{tr}(\epsilon^{-1})$ are too large, they are omitted here since they were not obtained with the precision used. (These large units are only missing from Table 1 for some cases of $h = 1$ or 3 when $\zeta_k/\zeta(1)$ is relatively large because one or more small primes split in the field. The first units missing are those for $m = 919$ which has $h = 1$ and both 2 and 3 as splitting primes.)

This table, and those that follow, were computed by a new, interesting method described in Marie Gras's paper [1]. The tables are more easily extended to larger m by this method if h is large. There are known criteria for $9|h$ and $4|h$, [2], [3]. Table 2 continues with 154 more $m < 10^4$ having $9|h$ while Table 3 contains 119 $m < 10^4$ having $4|h$. These two tables overlap some. Sometimes, units are missing, as before.